

STINESPRING'S THEOREM FOR MAPS ON HILBERT C^* -MODULES

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ABSTRACT. We strengthen Mohammad B. Asadi's analogue of Stinespring's theorem for certain maps on Hilbert C^* -modules. We also show that any two minimal Stinespring representations are unitarily equivalent. We illustrate the main theorem with an example.

1. INTRODUCTION

Stinespring's representation theorem is a fundamental theorem in the theory of completely positive maps. It is a structure theorem for completely positive maps from a C^* -algebra into the algebra of bounded operators on a Hilbert space. This theorem provides a representation for completely positive maps, showing that they are simple modifications of $*$ -homomorphisms (see [5] for details). One may consider it as a natural generalization of the well-known Gelfand-Naimark-Segal theorem for states on C^* -algebras (see [2, Theorem 4.5.2, page 278] for details). Recently, a theorem which looks like Stinespring's theorem was presented by Mohammad B. Asadi in [1] for a class of unital maps on Hilbert C^* -modules. Here we strengthen this result by removing a technical condition of Asadi's theorem [1]. We also remove the assumption of unitality on maps under consideration. Further we prove uniqueness up to unitary equivalence for minimal representations, which is an important ingredient of structure theorems like GNS theorem and Stinespring's theorem. Now the result looks even more like Stinespring's theorem.

1.1. Notation and Earlier Results. We denote Hilbert spaces by H, H_1, H_2 etc and the corresponding inner product and the induced norm by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Throughout we assume that the inner product is conjugate linear in the first variable and linear in the second variable. The space of bounded linear operators from H_1 to H_2 is denoted by $\mathcal{B}(H_1, H_2)$ and $\mathcal{B}(H, H) = \mathcal{B}(H)$. We denote C^* -algebras by \mathcal{A}, \mathcal{B} etc. The C^* -algebra of all $n \times n$ matrices with entries from \mathcal{A} is denoted by $\mathcal{M}_n(\mathcal{A})$. If L is a subset of a Hilbert space, then $[L] := \overline{\text{span}}(L)$.

A linear map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is said to be **positive** if $\phi(a^*a) \geq 0$, for all $a \in \mathcal{A}$. If $\phi_n : \mathcal{M}_n(\mathcal{A}) \rightarrow \mathcal{M}_n(\mathcal{B})$, given by $\phi_n((a_{ij})) = (\phi(a_{ij}))$, $i, j = 1, 2, \dots, n$ is positive, then ϕ is said to be n -positive. If ϕ_n is positive for all n ($n \geq 1$), then ϕ is called a **completely positive map**. Completely positive maps from a C^* -algebra

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\mathcal{A} to $\mathcal{B}(H)$ is characterized by Stinespring in [5]. This fundamental theorem is well-known as Stinespring's representation theorem.

Theorem 1.2. (*Stinespring's representation theorem* [4, Theorem 4.1, page 43]). *Let \mathcal{A} be a unital C^* -algebra and $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$ be a completely positive map. Then there exists a Hilbert space K , a unital $*$ -homomorphism $\rho : \mathcal{A} \rightarrow \mathcal{B}(K)$ and a bounded operator $V : H \rightarrow K$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^* \rho(a) V, \quad \text{for all } a \in \mathcal{A}.$$

The triple (ρ, V, K) in the Stinespring's representation theorem is called a representation for ϕ . If $[\rho(\mathcal{A})VH] = K$, then it is called a minimal representation. It is known that if (ρ, V, K) and (ρ', V', K') are two minimal Stinespring representations for ϕ , then there exists a unitary operator $U : K \rightarrow K'$ such that $UV = V'$ and $U\rho(a)U^* = \rho'(a)$ for all $a \in \mathcal{A}$ (see [4, Proposition 4.2, page 46]).

Now we consider maps on Hilbert C^* -modules. Let E be a Hilbert C^* -module over a C^* algebra \mathcal{A} (see [3] for details of Hilbert C^* -modules). Let $\phi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be linear. Then ϕ is said to be a morphism if it is a $*$ homomorphism and nondegenerate (i.e., $\overline{\phi(\mathcal{A})H_1} = H_1$). We remind the reader that $\mathcal{B}(H_1, H_2)$ is a Hilbert $\mathcal{B}(H_1)$ -module for any two Hilbert spaces H_1, H_2 , with the following operations:

- (1) module map: $(T, S) \mapsto TS : \mathcal{B}(H_1, H_2) \times \mathcal{B}(H_1) \rightarrow \mathcal{B}(H_1, H_2)$
- (2) inner product: $\langle T, S \rangle \mapsto T^*S : \mathcal{B}(H_1, H_2) \times \mathcal{B}(H_1, H_2) \rightarrow \mathcal{B}(H_1)$

A map $\Phi : E \rightarrow \mathcal{B}(H_1, H_2)$ is said to be a

- (1) ϕ -**map** if $\langle \Phi(x), \Phi(y) \rangle = \phi(\langle x, y \rangle)$ for all $x, y \in E$;
- (2) ϕ -**morphism** if Φ is a ϕ -map and ϕ is a morphism;
- (3) ϕ -**representation** if Φ is a ϕ -morphism and ϕ is a representation.

Note that a ϕ -morphism Φ is linear and satisfies $\Phi(xa) = \Phi(x)\phi(a)$ for every $x \in E$ and $a \in \mathcal{A}$.

Theorem 1.3. (*Mohammad B. Asadi* [1]). *If E is a Hilbert C^* -module over the unital C^* -algebra \mathcal{A} , and $\phi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ is a completely positive map with $\phi(1) = 1$ and $\Phi : E \rightarrow \mathcal{B}(H_1, H_2)$ is a ϕ -map with the additional property $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$ for some $x_0 \in E$, where H_1, H_2 are Hilbert spaces, then there exist Hilbert spaces K_1, K_2 and isometries $V : H_1 \rightarrow K_1$ and $W : H_2 \rightarrow K_2$ and a $*$ -homomorphism $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$ and a ρ -representation $\Psi : E \rightarrow \mathcal{B}(K_1, K_2)$ such that*

$$\phi(a) = V^* \rho(a) V, \quad \Phi(x) = W^* \Psi(x) V$$

for all $x \in E$, $a \in \mathcal{A}$.

The proof of this Theorem as given in [1] is erroneous as the sesquilinear form defined there on $E \otimes H_2$ is not positive definite. This can be fixed by interchanging the indices i, j in the definition of this form. However such a modification yields a 'non-minimal' representation. Moreover, the technical condition to have $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$ for some $x_0 \in E$ is completely unnecessary.

2. MAIN RESULTS

In this Section we strengthen Asadi's theorem for a ϕ -map Φ and discuss the minimality of the representations.

Theorem 2.1. *Let \mathcal{A} be a unital C^* -algebra and $\phi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be a completely positive map. Let E be a Hilbert \mathcal{A} -module and $\Phi : E \rightarrow \mathcal{B}(H_1, H_2)$ be a ϕ -map. Then there exists a pair of triples $((\rho, V, K_1), (\Psi, W, K_2))$, where*

- (1) K_1 and K_2 are Hilbert spaces;
- (2) $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$ is a unital $*$ -homomorphism and $\Psi : E \rightarrow \mathcal{B}(K_1, K_2)$ is a ρ -morphism;
- (3) $V : H_1 \rightarrow K_1$ and $W : H_2 \rightarrow K_2$ are bounded linear operators;

such that

$$\phi(a) = V^* \rho(a) V, \text{ for all } a \in \mathcal{A} \text{ and } \Phi(x) = W^* \Psi(x) V, \text{ for all } x \in E.$$

Proof. We prove the theorem in two steps.

Step I: Existence of ρ, V and K_1 : This is the content of Stinespring's theorem [4, Theorem 4.1, page 43] as ϕ is a completely positive map. In fact we can choose a minimal Stinespring representation for ϕ . In this case $K_1 = [\rho(\mathcal{A})VH_1]$.

Step II: Construction of Ψ, W and K_2 : Let $K_2 := [\Phi(E)H_1]$. Now define $\Psi : E \rightarrow \mathcal{B}(K_1, K_2)$ as follows:

For $x \in E$, define $\Psi(x) : K_1 \rightarrow K_2$ by

$$\Psi(x) \left(\sum_{j=1}^n \rho(a_j) V h_j \right) := \sum_{j=1}^n \Phi(x a_j) h_j, \quad a_j \in \mathcal{A}, h_j \in H_1, j = 1, \dots, n, n \geq 1.$$

First we claim that $\Psi(x)$ is well defined. Let $a_j \in \mathcal{A}, h_j \in H_1, j = 1, 2, \dots, n, n \geq 1$. Then we have

$$\begin{aligned} \|\Psi(x) \left(\sum_{j=1}^n \rho(a_j) V h_j \right)\|^2 &= \left\langle \sum_{j=1}^n \Phi(x a_j) h_j, \sum_{i=1}^n \Phi(x a_i) h_i \right\rangle \\ &= \sum_{i,j=1}^n \langle \Phi(x a_j) h_j, \Phi(x a_i) h_i \rangle \\ &= \sum_{i,j=1}^n \langle h_j, (\Phi(x a_j))^* \Phi(x a_i) h_i \rangle \\ &= \sum_{i,j=1}^n \langle h_j, \phi(a_j^* \langle x, x \rangle a_i) h_i \rangle \\ &= \sum_{i,j=1}^n \langle h_j, V^* \rho(a_j^* \langle x, x \rangle a_i) V h_i \rangle \\ &= \sum_{i,j=1}^n \langle \rho(a_j) V h_j, \rho(\langle x, x \rangle) \rho(a_i) V h_i \rangle \\ &= \left\langle \sum_{j=1}^n \rho(a_j) V h_j, \rho(\langle x, x \rangle) \left(\sum_{i=1}^n \rho(a_i) V h_i \right) \right\rangle \\ &\leq \|\rho(\langle x, x \rangle)\| \left\| \sum_{j=1}^n \rho(a_j) V h_j \right\|^2 \\ &\leq \|x\|^2 \left\| \sum_{j=1}^n \rho(a_j) V h_j \right\|^2. \end{aligned}$$

Hence $\Psi(x)$ is well defined and bounded. Hence it can be extended to whole of K_1 .

Next we prove that Ψ is a ρ -morphism. For this, let $x, y \in E, a_i, b_j \in \mathcal{A}, g_i, h_j \in H_1, i = 1, 2, \dots, m; j = 1, 2, \dots, n; m, n \geq 1$. Then

$$\begin{aligned}
& \left\langle \Psi(x)^* \Psi(y) \left(\sum_{j=1}^n \rho(b_j) V h_j \right), \sum_{i=1}^m \rho(a_i) V g_i \right\rangle \\
&= \left\langle \sum_{j=1}^n \Phi(y b_j) h_j, \sum_{i=1}^m \Phi(x a_i) g_i \right\rangle \\
&= \sum_{j=1}^n \sum_{i=1}^m \langle (\Phi(x a_i))^* \Phi(y b_j) h_j, g_i \rangle \\
&= \sum_{j=1}^n \sum_{i=1}^m \langle \phi(\langle x a_i, y b_j \rangle) h_j, g_i \rangle \\
&= \sum_{j=1}^n \sum_{i=1}^m \langle V^* \rho(a_i)^* \rho(\langle x, y \rangle) \rho(b_j) V h_j, g_i \rangle \\
&= \left\langle \rho(\langle x, y \rangle) \left(\sum_{j=1}^n \rho(b_j) V h_j \right), \sum_{i=1}^m \rho(a_i) V g_i \right\rangle.
\end{aligned}$$

Thus $\Psi(x)^* \Psi(y) = \rho(\langle x, y \rangle)$ on the dense set $\text{span}(\rho(A) V H_1)$ and hence they are equal on K_1 .

Note that $K_2 \subseteq H_2$. Let $W := P_{K_2}$, the orthogonal projection onto K_2 . Then $W^* : K_2 \rightarrow H_2$ is the inclusion map. Hence $W W^* = I_{K_2}$. That is W is a co-isometry.

Finally we give a representation for Φ . For $x \in E$ and $h \in H_1$, we have

$$W^* \Psi(x) V h = \Psi(x) V h = \Psi(x) (\rho(1) V h) = \Phi(x) h. \quad \square$$

Definition 2.2. Let ϕ and Φ be as in Theorem 2.1. We say that a pair of triples $((\rho, V, K_1), (\Psi, W, K_2))$ is a **Stinespring representation** of (ϕ, Φ) if conditions (1)-(3) of Theorem 2.1 are satisfied. Such a representation is said to be **minimal** if

- (a) $K_1 = [\rho(A) V H_1]$ and
- (b) $K_2 = [\Psi(E) V H_1]$.

Remark 2.3. Let ϕ and Φ be as in Theorem 2.1. The pair $((\rho, V, K_1), (\Psi, W, K_2))$ obtained in the proof of Theorem 2.1, is a minimal representation for (ϕ, Φ) .

Theorem 2.4. Let ϕ and Φ be as in Theorem 2.1. Assume that $((\rho, V, K_1), (\Psi, W, K_2))$ and $((\rho', V', K'_1), (\Psi', W', K'_2))$ are minimal representations for (ϕ, Φ) . Then there exists unitary operators $U_1 : K_1 \rightarrow K'_1$ and $U_2 : K_2 \rightarrow K'_2$ such that

- (1) $U_1 V = V', U_1 \rho(a) = \rho'(a) U_1$, for all $a \in \mathcal{A}$ and
- (2) $U_2 W = W', U_2 \Psi(x) = \Psi'(x) U_1$, for all $x \in E$.

That is, the following diagram commutes, for $a \in \mathcal{A}$ and $x \in E$:

$$\begin{array}{ccccccc}
H_1 & \xrightarrow{V} & K_1 & \xrightarrow{\rho(a)} & K_1 & \xrightarrow{\Psi(x)} & K_2 \xleftarrow{W} H_2 \\
& \searrow V' & \downarrow U_1 & & \downarrow U_1 & & \downarrow U_2 \swarrow W' \\
& & K'_1 & \xrightarrow{\rho'(a)} & K'_1 & \xrightarrow{\Psi'(x)} & K'_2
\end{array}$$

Proof. Existence of the unitary map $U_1 : K_1 \rightarrow K'_1$ follows by Theorem [4, Theorem 4.2, page 46]. This can be obtained as follows: First define $U_1 : \text{span}(\rho(\mathcal{A})VH_1) \rightarrow \text{span}(\rho'(\mathcal{A})V'H_1)$ by

$$U_1\left(\sum_{j=1}^n \rho(a_j)Vh_j\right) := \sum_{j=1}^n \rho'(a_j)V'h_j, \quad a_j \in \mathcal{A}, h_j \in H_1, j = 1, \dots, n, n \geq 1,$$

which can be seen to be an onto isometry. Let us denote the extension of U_1 to K_1 by U_1 itself. Then U_1 is unitary and satisfies the conditions in (1).

Now define $U_2 : \text{span}(\Psi(E)VH_1) \rightarrow \text{span}(\Psi'(E)V'H_1)$ by

$$U_2\left(\sum_{j=1}^n \Psi(x_j)Vh_j\right) := \sum_{j=1}^n \Psi'(x_j)V'h_j, \quad x_j \in E, h_j \in H_1, j = 1, 2, \dots, n, n \geq 1.$$

We claim that U_2 is well defined and can be extended to a unitary map. For this consider

$$\begin{aligned} \left\| \sum_{j=1}^n \Psi'(x_j)V'h_j \right\|^2 &= \left\langle \sum_{j=1}^n \Psi'(x_j)V'h_j, \sum_{i=1}^n \Psi'(x_i)V'h_i \right\rangle \\ &= \sum_{i,j=1}^n \langle \Psi'(x_j)V'h_j, \Psi'(x_i)V'h_i \rangle \\ &= \sum_{i,j=1}^n \langle h_j, V'^*(\Psi'(x_j))^* \Psi'(x_i)V'h_i \rangle \\ &= \sum_{i,j=1}^n \langle h_j, V'^* \rho'(\langle x_j, x_i \rangle) V'h_i \rangle \\ &= \sum_{i,j=1}^n \langle h_j, V^* \rho(\langle x_j, x_i \rangle) Vh_i \rangle \\ &= \sum_{i,j=1}^n \langle h_j, V^*(\Psi(x_j))^* \Psi(x_i)Vh_i \rangle \\ &= \left\langle \sum_{j=1}^n \Psi(x_j)Vh_j, \sum_{i=1}^n \Psi(x_i)Vh_i \right\rangle \\ &= \left\| \sum_{j=1}^n \Psi(x_j)Vh_j \right\|^2. \end{aligned}$$

This concludes that U_2 is well defined and an isometry. Hence it can be extended to whole of K_2 , call the extension U_2 itself, and being onto it is a unitary.

Since $((\rho, V, K_1), (\Psi, W, K_2))$ and $((\rho', V', K'_1), (\Psi', W', K'_2))$ are representations for (ϕ, Φ) , it follows that

$$\begin{aligned} \Phi(x) &= W^* \Psi(x)V = W'^* \Psi'(x)V' = W'^* U_2 \Psi(x)V \\ &\Rightarrow (W^* - W'^* U_2) \Psi(x)V = 0, \end{aligned}$$

equivalently, $(W^* - W'^* U_2) \Psi(x)Vh = 0$ for all $h \in H_1$. Since $[\Psi(E)VH_1] = K_2$, it follows that $U_2 W = W'$.

To prove the remaining part of (2), it is enough to show $U_2\Psi(x) = \Psi'(x)U_1$ on the dense set $\text{span}(\rho(\mathcal{A})VH_1)$. Let $a_j \in \mathcal{A}, h_j \in H_1, j = 1, 2, \dots, n, n \geq 1$. Consider

$$\begin{aligned}
U_2\Psi(x)\left(\sum_{j=1}^n \rho(a_j)Vh_j\right) &= U_2\left(\sum_{j=1}^n \Psi(xa_j)Vh_j\right) \quad (\text{since } \Psi \text{ is } \rho\text{-morphism}) \\
&= \sum_{j=1}^n \Psi'(xa_j)V'h_j \\
&= \Psi'(x)\left(\sum_{j=1}^n \rho'(a_j)V'h_j\right) \quad (\text{since } \Psi' \text{ is } \rho'\text{-morphism}) \\
&= \Psi'(x)U_1\left(\sum_{j=1}^n \rho(a_j)Vh_j\right). \quad \square
\end{aligned}$$

Remark 2.5. Let $((\rho, V, K_1), (\Psi, W, K_2))$ be a Stinespring representation for (ϕ, Φ) . If ϕ is unital, then V is an isometry. If the representation is minimal, then W is a co-isometry by the proof of Theorem 2.1 and (2) of Theorem 2.4.

We give an example to illustrate our result.

Example 2.6. Let $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$, $H_1 = \mathbb{C}^2, H_2 = \mathbb{C}^8$ and $E = \mathcal{A} \oplus \mathcal{A}$. Define $\phi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ by

$$\phi\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & \frac{a_{12}}{2} \\ \frac{a_{21}}{2} & a_{22} \end{pmatrix}, \text{ for all } a_{ij} \in \mathbb{C}, i, j = 1, 2.$$

Let $D = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$. Then $\phi(A) = D \circ A$, for all $A \in \mathcal{A}$, here \circ denote the Schur product. As D is positive, ϕ is a completely positive map (see [4, Theorem 3.7, page 31] for details). Define $\Phi : E \rightarrow \mathcal{B}(H_1, H_2)$ by

$$\Phi((a_{ij} \oplus (b_{ij}))) = \begin{pmatrix} \frac{\sqrt{3}}{2}a_{11} & \frac{\sqrt{3}}{2}a_{12} \\ \frac{\sqrt{3}}{2}a_{21} & \frac{\sqrt{3}}{2}a_{22} \\ \frac{\sqrt{3}}{2}b_{11} & \frac{\sqrt{3}}{2}b_{12} \\ \frac{\sqrt{3}}{2}b_{21} & \frac{\sqrt{3}}{2}b_{22} \\ \frac{1}{2}a_{11} & -\frac{1}{2}a_{12} \\ \frac{1}{2}a_{21} & -\frac{1}{2}a_{22} \\ \frac{1}{2}b_{11} & -\frac{1}{2}b_{12} \\ \frac{1}{2}b_{21} & -\frac{1}{2}b_{22} \end{pmatrix}, \quad (a_{ij}), (b_{ij}) \in \mathcal{A}, i, j = 1, 2.$$

It can be verified that Φ is a ϕ -map.

Let $K_1 = \mathbb{C}^4$ and $K_2 = H_2$. In this case $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$, $V : H_1 \rightarrow K_1$ and $\Psi : E \rightarrow \mathcal{B}(K_1, K_2)$ are given by

$$\begin{aligned}\rho((a_{ij})) &= \begin{pmatrix} (a_{ij}) & 0 \\ 0 & (a_{ij}) \end{pmatrix}, \quad (a_{ij}) \in \mathcal{A}, \quad i, j = 1, 2. \\ V &= \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\ \Psi((a_{ij}) \oplus (b_{ij})) &= \begin{pmatrix} (a_{ij}) & 0 \\ (b_{ij}) & 0 \\ 0 & (a_{ij}) \\ 0 & (b_{ij}) \end{pmatrix}, \quad (a_{ij}), (b_{ij}) \in \mathcal{A}, \quad i, j = 1, 2.\end{aligned}$$

It is easy to verify that Ψ is a ρ -morphism and $\Phi((a_{ij}) \oplus (b_{ij})) = W^* \Psi((a_{ij}) \oplus (b_{ij}))V$, where $W = I_{H_2}$. This example illustrate Theorem (2.1).

Remark 2.7. Note that in Example 2.6, there does not exists an $x_0 \in E$ with the property that $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$, which is an assumption in Theorem 1.3.

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